Measurement and Geometry

- Relate the geometric representation and the algebraic representation, \( a^2 + b^2 = c^2 \), of the Pythagorean theorem.
- Solve problems using the Pythagorean theorem.
- Solve problems involving the areas and perimeters of composite shapes.
- Determine, through investigation, the formula for the surface area of a pyramid.
- Develop, through investigation, the formulas for the volume of a pyramid, a cone, and a sphere.
- Solve problems involving the surface areas and volumes of prisms, pyramids, cylinders, cones, and spheres.

Vocabulary

- hypotenuse
- Pythagorean theorem
- surface area
- volume
- pyramid
- lateral faces
- cone
- sphere

Measurement Relationships

Landscape architects design, plan, and manage land. Their work can be seen in the attractiveness and usefulness of parks, highways, neighbourhoods, gardens, and zoos. Concepts of measurement in two-dimensional and three-dimensional geometry are important in landscape design and construction.

In this chapter, you will solve problems involving two-dimensional and three-dimensional figures. You will also extend your skills with three-dimensional geometry to include pyramids, cones, and spheres.
Chapter Problem

For her summer job, Emily will help her brother with his landscaping business. The company designs and installs patios and gardens and puts the finishing touches around swimming pools. Throughout this chapter, you will apply your skills to help Emily complete her projects.
Calculate Perimeter and Circumference

The perimeter of a shape is the distance around the outside. Circumference is the perimeter of a circle.

\[ P = 2(l + w) \]
\[ = 2(8.2 + 5.6) \]
\[ = 2(13.8) \]
\[ = 27.6 \]

The perimeter of the rectangle is 27.6 cm.

\[ C = 2\pi r \]
\[ = 2\pi (5.3) \]

Estimate: \(2 \times 3 \times 5 = 30\)

\[ \approx 33.3 \]

The circumference of the circle is approximately 33.3 cm.

1. Determine the perimeter of each shape.
   a) \[
   \begin{array}{c}
   4 \text{ m} \\
   0.8 \text{ m}
   \end{array}
   \]
   b) \[
   \begin{array}{c}
   6.5 \text{ cm}
   \end{array}
   \]
   c) \[
   \begin{array}{c}
   2.1 \text{ mm}
   \end{array}
   \]
   d) \[
   \begin{array}{c}
   2.2 \text{ cm}
   \end{array}
   \]
   e) \[
   \begin{array}{c}
   15 \text{ m} \\
   30 \text{ m}
   \end{array}
   \]
   f) \[
   \begin{array}{c}
   5 \text{ mm} \\
   7.5 \text{ mm}
   \end{array}
   \]

2. Determine the circumference of each circle. Round answers to the nearest tenth of a unit.
   a) \[
   \begin{array}{c}
   2.8 \text{ cm}
   \end{array}
   \]
   b) \[
   \begin{array}{c}
   10.2 \text{ m}
   \end{array}
   \]
   c) \[
   \begin{array}{c}
   35 \text{ mm}
   \end{array}
   \]
   d) \[
   \begin{array}{c}
   12.5 \text{ cm}
   \end{array}
   \]

3. A flower bed has the dimensions shown.
   \[
   \begin{array}{c}
   9 \text{ m} \\
   6 \text{ m}
   \end{array}
   \]
   \[
   \begin{array}{c}
   10 \text{ m} \\
   17 \text{ m}
   \end{array}
   \]
   Find the perimeter of the flower bed.
Apply Area Formulas

Area measures how much space a two-dimensional shape covers. It is measured in square units.

The table gives the area formulas for some common shapes.

<table>
<thead>
<tr>
<th>Shape</th>
<th>Name</th>
<th>Area Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image" alt="Rectangle" /></td>
<td>rectangle</td>
<td>( A = lw )</td>
</tr>
<tr>
<td><img src="image" alt="Triangle" /></td>
<td>triangle</td>
<td>( A = \frac{1}{2}bh )</td>
</tr>
<tr>
<td><img src="image" alt="Circle" /></td>
<td>circle</td>
<td>( A = \pi r^2 )</td>
</tr>
<tr>
<td><img src="image" alt="Parallelogram" /></td>
<td>parallelogram</td>
<td>( A = bh )</td>
</tr>
<tr>
<td><img src="image" alt="Trapezoid" /></td>
<td>trapezoid</td>
<td>( A = \frac{1}{2}h(a + b) )</td>
</tr>
</tbody>
</table>

Apply the formula for the area of a rectangle. Substitute \( l = 6.0 \) and \( w = 4.5 \).

\[
A = lw
= (6.0)(4.5)
= 27
\]

The area of the rectangle is 27 cm².

4. Determine the area of each shape. Round answers to the nearest tenth of a square unit.

5. Determine the area of each shape.

a) ![Triangle](image) \( 7.5 \text{ cm} \)

b) ![Circle](image) \( 5.8 \text{ m} \)

a) ![Parallelogram](image) \( 2.1 \text{ m} \)

b) ![Trapezoid](image) \( 8.4 \text{ cm} \)
Calculate Surface Area and Volume

A net, which is a flat pattern that can be folded to form a figure, can help you visualize the faces of a three-dimensional figure.

Apply the formula for the surface area of a cylinder.

\[
SA = 2\pi r^2 + 2\pi rh
\]

\[
= 2\pi (4)^2 + 2\pi (4)(20)
\]

\[
= 2\pi \times 16 + 2\pi \times 80
\]

\[
= 16\pi + 160\pi
\]

\[
= 176\pi
\]

\[
\approx 548.8
\]

The surface area is approximately 548.8 cm².

Apply the formula for the volume of a cylinder.

\[
V = \text{(area of base)}(\text{height})
\]

\[
= \pi r^2 h
\]

\[
= \pi (4)^2 (20)
\]

\[
= \pi \times 16 \times 20
\]

\[
= 320\pi
\]

\[
\approx 1005
\]

The volume is approximately 1005 cm³.

6. Determine the surface area of each three-dimensional figure. If necessary, round answers to the nearest square unit.

a) b)

7. Find the volume of each three-dimensional figure in question 6. If necessary, round answers to the nearest cubic unit.

8. a) Draw a net for the triangular prism. What shapes do you need? Label the dimensions on the shapes in your net.

b) Find the surface area of the prism.

c) Find the volume of the prism.
Use *The Geometer’s Sketchpad®*

Draw and measure a line segment using *The Geometer’s Sketchpad®*.

- Use the **Straightedge Tool** to create a line segment AB.
- Use the **Selection Arrow Tool** to select the line segment.
- From the **Measure** menu, choose **Length**.

Drag one of the endpoints of line segment AB to change its length. Notice how the measurement on the screen changes as you do this.

Draw and measure the perimeter and area of a triangle using *The Geometer’s Sketchpad®*.

- Use the **Straightedge Tool** to create three line segments to form ΔABC.
- Use the **Selection Arrow Tool** to select all three vertices.
- From the **Construct** menu, choose **Triangle Interior**.
- From the **Measure** menu, choose **Perimeter** and then **Area**.

Drag one vertex of the triangle to change its shape. Notice how the perimeter and area measurements change as you drag the vertex.

Draw and measure the circumference and area of a circle using *The Geometer’s Sketchpad®*.

- Use the **Compass Tool** to create any circle.
- Make sure the circle is selected. Then, from the **Measure** menu, choose **Circumference** and then **Area**.

You can also measure the radius.

- Select the circle. From the **Measure** menu, choose **Radius**.

Change the size of the circle and watch the measurements change.

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**9.** Use *The Geometer’s Sketchpad®* to create a triangle with each characteristic.

a) a perimeter of 15 cm

b) an area of 10 cm²

**10.** Use *The Geometer’s Sketchpad®* to create a circle with each characteristic.

a) a circumference of 12 cm

b) an area of 20 cm²

**11.** Use *The Geometer’s Sketchpad®* to create any circle.

a) Measure its circumference and area.

b) Create a quadrilateral that has the same perimeter. Predict which figure has the greater area.

c) Calculate the area of the quadrilateral. Was your prediction correct?
Apply the Pythagorean Theorem

The Pythagorean theorem is named after the Greek philosopher and mathematician Pythagoras (580–500 B.C.E.). Although ancient texts indicate that different civilizations understood this property of right triangles, Pythagoras proved that it applies to all right triangles.

If a right triangle is labelled as shown, then the area of the large square drawn on the hypotenuse is $c^2$, while the areas of the other two squares are $a^2$ and $b^2$.

According to the Pythagorean relationship, the area of the square drawn on the hypotenuse is equal to the sum of the areas of the squares drawn on the other two sides.

Therefore, the algebraic model for the Pythagorean relationship is $c^2 = a^2 + b^2$. This is known as the **Pythagorean theorem**.

**Investigate**

**How can you illustrate the Pythagorean theorem?**

**Method 1: Use Pencil and Paper**

1. Construct any right triangle. Label the sides of your triangle using three different letters.

2. Measure the length of each side of your triangle. Indicate these measures on your diagram.

3. a) Calculate the area of the square on the hypotenuse.
   
   b) Calculate the sum of the areas of the squares on the two shorter sides.

   c) Write the Pythagorean theorem using your side labels.
4. a) Calculate the square root of your answer to step 3b).
   b) Compare this value to the length of the hypotenuse.

5. Construct any non-right triangle. Does the Pythagorean relationship still hold? Does the relationship from step 4, part b), still hold?

6. Reflect Explain how this activity illustrates the Pythagorean theorem.

Method 2: Use The Geometer's Sketchpad®

1. From the Edit menu, choose Preferences. Click on the Units tab. Set the precision to tenths for all three boxes. Click on the Text tab and check For All New Points. Click on OK.

2. Use the Straightedge Tool to create any ΔABC.

3. a) To measure ∠ABC, select vertices A, B, and C, in that order. From the Measure menu, choose Angle.
   b) To measure the length of AB, select line segment AB. From the Measure menu, choose Length. Repeat for line segments BC and CA.

4. a) Drag a vertex of the triangle until ∠ABC measures 90°.
   b) Select the measure \( m \overrightarrow{CA} \). From the Measure menu, choose Calculate. Enter \( m \overrightarrow{CA}^2 \), by selecting \( m \overrightarrow{CA} \) from the Values drop-down menu on the calculator.
   c) Select \( m \overrightarrow{AB} \) and \( m \overrightarrow{BC} \). From the Measure menu, choose Calculate. Enter \( m \overrightarrow{AB}^2 + m \overrightarrow{BC}^2 \).

8.1 Apply the Pythagorean Theorem • MHR 419
5. a) Select \( (m\overline{AB})^2 \) and \( (m\overline{BC})^2 \). From the Measure menu, choose Calculate. Evaluate \( \sqrt{(m\overline{AB})^2 + (m\overline{BC})^2} \) by choosing sqrt from the Functions pull-down menu on the calculator.

b) Compare this value to the length of side CA.

6. Drag a vertex of the triangle so that the measure of \( \angle ABC \) is no longer 90°. Does the Pythagorean relationship still hold? Does the relationship from step 5b) still hold?

7. Reflect Explain how this activity illustrates the Pythagorean theorem.

Example 1  Find the Hypotenuse

The advertised size of a computer or television screen is actually the length of the diagonal of the screen. A computer screen measures 30 cm by 22.5 cm. Determine the length of its diagonal.

Solution

In the diagram, the diagonal, \( d \), is the hypotenuse.

Apply the Pythagorean theorem.

\[
d^2 = 30^2 + 22.5^2 \\
d^2 = 900 + 506.25 \\
d^2 = 1406.25 \\
\sqrt{d^2} = \sqrt{1406.25} \quad \text{Only the positive square root needs to be used because} \\
d = 37.5 \quad \text{\( d \) is a length.}
\]

The length of the diagonal of the computer screen is 37.5 cm.

Example 2  Find One of the Shorter Sides

Jenna is changing a light bulb. She rests a 4-m ladder against a vertical wall so that its base is 1.4 m from the wall. How high up the wall does the top of the ladder reach? Round your answer to the nearest tenth of a metre.
Solution

In this case, the ladder is the hypotenuse, with a length of 4 m. The unknown side length is \( h \).

Apply the Pythagorean theorem.

\[
4^2 = 1.4^2 + h^2
\]
\[
16 = 1.96 + h^2
\]
\[
16 - 1.96 = 1.96 - 1.96 + h^2 \quad \text{Subtract 1.96 from both sides.}
\]
\[
14.04 = h^2
\]
\[
\sqrt{14.04} = \sqrt{h^2} \quad \text{Take the square root of both sides.}
\]
\[
3.7 = h
\]

The ladder reaches 3.7 m up the wall, to the nearest tenth of a metre.

Example 3 Calculate the Area of a Right Triangle

Calculate the area of the triangular sail on the toy sailboat.

Solution

The formula for the area of a triangle is \( A = \frac{1}{2}bh \).

The base, \( b \), and the height, \( h \), must be perpendicular to each other. For a right triangle, the base and the height are the lengths of the two shorter sides.

First, use the Pythagorean theorem to find the length of the unknown side, \( a \).

\[
11^2 = a^2 + 8^2
\]
\[
121 = a^2 + 64
\]
\[
121 - 64 = a^2 + 64 - 64 \quad \text{Subtract 64 from both sides.}
\]
\[
57 = a^2
\]
\[
\sqrt{57} = \sqrt{a^2} \quad \text{Take the square root of both sides.}
\]
\[
a \approx 7.5
\]

The length of side \( a \) is approximately 7.5 cm.

Now, apply the formula for the area of a triangle.

\[
A = \frac{1}{2}bh
\]
\[
= \frac{1}{2}(8)(7.5)
\]
\[
= 30
\]

The area of the sail is approximately 30 cm\(^2\).
Key Concepts

- The longest side of a right triangle is the hypotenuse.
- The Pythagorean theorem states that the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the two shorter sides.
- An algebraic model representing the Pythagorean theorem is \( c^2 = a^2 + b^2 \), where \( c \) represents the length of the hypotenuse and \( a \) and \( b \) represent the lengths of the two shorter sides.
- You can use the Pythagorean theorem to calculate the length of an unknown side of a right triangle.
- You can calculate the area of a right triangle by using the formula \( A = \frac{1}{2}bh \), with the lengths of the two shorter sides as the base, \( b \), and the height, \( h \). If one of these dimensions is unknown and you know the hypotenuse, apply the Pythagorean theorem to calculate the length of the unknown side. Then, use the area formula.

Communicate Your Understanding

C1 Describe how you can use the Pythagorean theorem to determine the length of the diagonal of the square.

C2 Describe how you can use the Pythagorean theorem to determine the distance between two points on a grid.

C3 Describe how you would find the area of a right triangle if you knew the lengths of the hypotenuse and one of the other two sides.
For help with question 1, see Example 1.

1. Calculate the length of the hypotenuse in each triangle. Round your answers to the nearest tenth of a unit, when necessary.

   a) \[
   \begin{align*}
   6 \text{ cm} & \quad \text{c} \\
   8 \text{ cm} & \quad \text{a}
   \end{align*}
   \]

   b) \[
   \begin{align*}
   12 \text{ m} & \quad \text{c} \\
   5 \text{ m} & \quad \text{b}
   \end{align*}
   \]

   c) \[
   \begin{align*}
   4.2 \text{ m} & \quad \text{c} \\
   5.1 \text{ m} & \quad \text{a}
   \end{align*}
   \]

   d) \[
   \begin{align*}
   7 \text{ cm} & \quad \text{c} \\
   5 \text{ cm} & \quad \text{a}
   \end{align*}
   \]

For help with question 2, see Example 2.

2. Calculate the length of the unknown side in each triangle. Round your answers to the nearest tenth of a unit, when necessary.

   a) \[
   \begin{align*}
   17 \text{ cm} & \quad \text{a} \\
   8 \text{ cm} & \quad \text{b}
   \end{align*}
   \]

   b) \[
   \begin{align*}
   4 \text{ m} & \quad \text{a} \\
   10 \text{ m} & \quad \text{b}
   \end{align*}
   \]

   c) \[
   \begin{align*}
   9.5 \text{ m} & \quad \text{a} \\
   5.5 \text{ m} & \quad \text{b}
   \end{align*}
   \]

   d) \[
   \begin{align*}
   8.2 \text{ cm} & \quad \text{a} \\
   3.6 \text{ cm} & \quad \text{b}
   \end{align*}
   \]

For help with question 3, see Example 3.

3. Determine the area of each right triangle. Round your answers to the nearest tenth of a square unit, when necessary.

   a) \[
   \begin{align*}
   8 \text{ cm} & \quad \text{a} \\
   10 \text{ cm} & \quad \text{b}
   \end{align*}
   \]

   b) \[
   \begin{align*}
   7 \text{ m} & \quad \text{a} \\
   12 \text{ m} & \quad \text{b}
   \end{align*}
   \]
**Connect and Apply**

4. Calculate the length of each line segment. Round answers to the nearest tenth of a unit, when necessary.

   a) AB
   
   b) CD
   
   c) EF

5. What is the length of the diagonal of a computer screen that measures 28 cm by 21 cm?

6. A baseball diamond is a square with sides that measure about 27 m. How far does the second-base player have to throw the ball to get a runner out at home plate? Round your answer to the nearest metre.

7. A square courtyard has diagonal paths that are each 42 m long. What is the perimeter of the courtyard, to the nearest metre?

8. Brook is flying a kite while standing 50 m from the base of a tree at the park. Her kite is directly above the 10-m tree and the 125-m string is fully extended. Approximately how far above the tree is her kite flying?

9. **Chapter Problem** Emily has designed a triangular flower bed for the corner of her client’s rectangular lot. The bed is fenced on two sides and Emily will use border stones for the third side. The bed measures 2 m and 2.5 m along the fenced sides. How many border stones, 30 cm in length, will Emily need to edge the flower bed?
10. A cardboard box measures 40 cm by 40 cm by 30 cm. Calculate the length of the space diagonal, to the nearest centimetre.

11. The Spider and the Fly Problem is a classic puzzle that originally appeared in an English newspaper in 1903. It was posed by H.E. Dudeney. In a rectangular room with dimensions 30 ft by 12 ft by 12 ft, a spider is located in the middle of one 12 ft by 12 ft wall, 1 ft away from the ceiling. A fly is in the middle of the opposite wall 1 ft away from the floor. If the fly does not move, what is the shortest distance that the spider can crawl along the walls, ceiling, and floor to capture the fly?
   Hint: Using a net of the room will help you get the answer, which is less than 42 ft!

12. A spiral is formed with right triangles, as shown in the diagram.
   a) Calculate the length of the hypotenuse of each triangle, leaving your answers in square root form. Describe the pattern that results.
   b) Calculate the area of the spiral shown.
   c) Describe how the expression for the area would change if the pattern continued.

13. Math Contest
   a) The set of whole numbers (5, 12, 13) is called a Pythagorean triple. Explain why this name is appropriate.
   b) The smallest Pythagorean triple is (3, 4, 5). Investigate whether multiples of a Pythagorean triple make Pythagorean triples.
   c) Substitute values for \( m \) and \( n \) to investigate whether triples of the form \( (m^2 - n^2, 2mn, m^2 + n^2) \) are Pythagorean triples.
   d) What are the restrictions on the values of \( m \) and \( n \) in part c)?
Investigate

How can you apply your knowledge of perimeter and area to a composite figure?

The owners of a restaurant have decided to build an outdoor patio to increase the number of customers that they can serve in the summer. The patio design consists of a rectangle, two right triangles, and a semicircle.

The patio area will be made of interlocking paving stones with different stones along the border. The paving stones cost $52.95/m². The border stones are priced according to the length of the border and cost $15.50/m. How much will the materials for the patio cost, including 8% PST and 7% GST? Allow an additional 10% to account for stones that must be cut for the design.

1. Before making any calculations, estimate the cost of the stones for the patio.

2. To calculate the perimeter of the patio, you will need to determine some of the outside measurements.
   a) Describe how you can calculate the perimeter of the semicircle and the lengths of the two unlabelled sides of the triangles.
   b) Calculate each of the unknown outside measurements.
   c) Calculate the total perimeter. Add 10% for waste due to cuts.
3. Now, consider the area of the patio.
   a) Describe the simple shapes that make up the area.
   b) Describe how you will calculate the area of each shape.
   c) Calculate the total area of the patio. Again, add 10% for waste.

4. a) Calculate the cost of the materials for the patio before taxes.
   b) Calculate the total cost of the materials, including 8% PST and 7% GST.
   c) Compare this answer to your original estimate. How close were you?

5. Reflect Describe an advantage to using simple shapes to calculate the perimeter and area of a composite figure.

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**Example 1 Area and Perimeter of a Composite Figure**

a) Determine the area of the stained-glass panel shown.

b) Determine the perimeter.
   Round to the nearest centimetre.

**Solution**

a) The stained-glass panel can be split into a rectangle and two right triangles.

To find the total area of the panel, add the area of the rectangle and the areas of the two right triangles. Use the formulas for the areas of these shapes.

Call the area of the rectangle $A_{\text{rectangle}}$.

\[
A_{\text{rectangle}} = lw \\
= (24)(16) \\
= 384
\]
Call the area of the triangle on the left $A_{\text{triangle 1}}$.

$$A_{\text{triangle 1}} = \frac{1}{2} bh$$

$$= \frac{1}{2} (12)(16)$$

$$= 96$$

Call the area of the triangle on the right $A_{\text{triangle 2}}$.

$$A_{\text{triangle 2}} = \frac{1}{2} bh$$

$$= \frac{1}{2} (4)(16)$$

$$= 32$$

Call the total area $A_{\text{total}}$.

$$A_{\text{total}} = A_{\text{rectangle}} + A_{\text{triangle 1}} + A_{\text{triangle 2}}$$

$$= 384 + 96 + 32$$

$$= 512$$

The total area of the stained-glass panel is 512 cm$^2$.

This stained-glass panel is in the shape of a trapezoid. Another way to calculate the area of this figure is to use the formula for a trapezoid, $A = \frac{1}{2} h(a + b)$ or $A = \frac{h(a + b)}{2}$.

$$A = \frac{1}{2} h(a + b)$$

$$= \frac{1}{2} (16)(24 + 40)$$

$$= 8(64)$$

$$= 512$$

The area of the stained-glass panel is 512 cm$^2$.

b) The perimeter of the stained-glass panel includes two unknown side lengths.

When the figure is split into a rectangle and two right triangles, each unknown side is in a triangle. Apply the Pythagorean theorem to determine the lengths of the two unknown sides in the perimeter.
First, find the length of the unknown side on the left. Call it \( c \).
\[
c^2 = 12^2 + 16^2
\]
\[
c^2 = 144 + 256
\]
\[
c^2 = 400
\]
\[
c = \sqrt{400}
\]
\[
c = 20
\]
Next, find the length of the unknown side on the right. Call it \( d \).
\[
d^2 = 4^2 + 16^2
\]
\[
d^2 = 16 + 256
\]
\[
d^2 = 272
\]
\[
d = \sqrt{272}
\]
\[
d \approx 16
\]
Now, find the perimeter by adding the outside measurements.
\[
P = 24 + 16 + 40 + 20
\]
\[
P = 100
\]
The perimeter of the stained-glass panel is approximately 100 cm.

The two unknown sides of the trapezoid must each be longer than 16 cm. This means that the total perimeter must be longer than \((24 + 16 + 40 + 16)\) or 96 cm. A perimeter of 100 cm seems reasonable for this stained-glass panel.

**Example 2** Area of a Composite Figure, by Subtraction, and Perimeter

A hotel is remodelling its outdoor entrance area. The new design includes a tile walkway leading to a semicircular fountain.

**a)** Describe the steps you would use to find the area of the walkway.

**b)** Calculate the area of the walkway. Round to the nearest tenth of a square metre.

**c)** The walkway will have a border in a different colour of tile. Calculate the perimeter of the walkway. Round to the nearest tenth of a metre.
**Solution**

a) The walkway is a rectangle with a semicircle cut out of it.

Determine the area of the rectangle minus the area of the semicircle.

\[ A_{\text{rectangle}} = lw \]
\[ = (5.2)(2.1) \]
\[ = 10.92 \]

A semicircle is half a circle. So, the area of a semicircle is \( \frac{1}{2} \) the area of a circle.

\[ A_{\text{semicircle}} = \frac{1}{2} \pi r^2 \]
\[ = \frac{1}{2} \pi (1.05)^2 \]
\[ = \frac{1}{2} \pi (1.05)(1.05) \]
\[ = \frac{1}{2} \pi 1.05 \]
\[ \div 1.73 \]

\[ A_{\text{walkway}} = A_{\text{rectangle}} - A_{\text{semicircle}} \]
\[ = 10.92 - 1.73 \]
\[ = 9.19 \]

The area of the walkway is approximately 9.2 m².

b) \( A_{\text{rectangle}} = lw \)
\[ = (5.2)(2.1) \]
\[ \text{Estimate: } 5 \times 2 = 10 \]

\[ A_{\text{semicircle}} = \frac{1}{2} \pi r^2 \]
\[ = \frac{1}{2} \pi (1.05)(1.05) \]
\[ \div 1.73 \]

\[ A_{\text{walkway}} = A_{\text{rectangle}} - A_{\text{semicircle}} \]
\[ = 10.92 - 1.73 \]
\[ = 9.19 \]

The area of the walkway is approximately 9.2 m².

c) The perimeter of the walkway consists of the three sides of the rectangular section and the semicircular arc.

First, find the length of the semicircular arc.

\[ L = \frac{1}{2} (\pi d) \]
\[ = \frac{1}{2} \pi (2.1) \]
\[ \div 3.3 \]

The formula for the circumference of a circle is \( C = \pi d \). So, the length of a semicircular arc is half the circumference.

Now, add the distances around the outside of the walkway.

\[ P_{\text{walkway}} = L + \text{three sides of rectangle} \]
\[ = 3.3 + (5.2 + 2.1 + 5.2) \]
\[ = 15.8 \]

The perimeter of the walkway is about 15.8 m.
Key Concepts

- A composite figure is made up of more than one simple shape.
- To determine the total area of a composite figure, add and/or subtract areas.
- To determine the perimeter of a composite figure, add the distances around the outside of the figure.

Communicate Your Understanding

C1 Refer to the Investigate on pages 426 and 427. The patio was divided into four simple shapes: a rectangle, two triangles, and a semicircle. Describe how to determine the area of the patio by adding the areas of two shapes.

C2 Consider the yard shown.

a) Describe how you can determine the unknown lengths.

b) Describe how you can determine the area of the yard by adding the areas of simpler shapes.

c) Describe how you can determine the area by subtracting areas.

C3 a) Suppose you need to calculate the perimeter of the yard in question C2. Explain why you cannot simply add the perimeters of the rectangles that make up the composite figure.

b) Without calculating, describe how the perimeter of this yard compares to the perimeter of a rectangular yard that measures 10 m by 7 m.

C4 a) How does the perimeter of the yard in question C2 compare to the perimeter of the yard shown, which has been increased by the smaller rectangular section rather than being decreased in size?

b) Describe how you would determine the area of this yard.

c) How does its area compare to the area of the yard in question C2?
**Practise**

*For help with questions 1 and 2, see Examples 1 and 2.*

1. For each composite figure,
   - solve for any unknown lengths
   - determine the perimeter
   
   Round to the nearest unit, if necessary.

   a) ![Composite Figure A](image)
   b) ![Composite Figure B](image)
   c) ![Composite Figure C](image)
   d) ![Composite Figure D](image)
   e) ![Composite Figure E](image)

2. Calculate the area of each composite figure.
   
   Round to the nearest square unit, if necessary.

   a) ![Composite Figure A](image)
   b) ![Composite Figure B](image)
   c) ![Composite Figure C](image)
   d) ![Composite Figure D](image)
   e) ![Composite Figure E](image)
   f) ![Composite Figure F](image)
**Connect and Apply**

3. **a)** What length of fencing is needed to surround this yard, to the nearest metre?
   **b)** What is the area of the yard?
   **c)** Explain the steps you took to solve this problem.

4. Patrick is planning a garage sale. He is painting six arrow signs to direct people to his sale.
   **a)** Calculate the area of one side of one arrow.
   **b)** Each can of paint can cover 2 m². How many cans of paint will Patrick need to paint all six signs?
   **c)** If the paint costs $3.95 per can, plus 8% PST and 7% GST, how much will it cost Patrick to paint the six signs?

5. Arif has designed a logo of her initial as shown. Use a ruler to make the appropriate measurements and calculate the area of the initial, to the nearest hundred square millimetres.

6. Create your own initial logo similar to the one in question 5. Calculate the total area of your logo.

7. **Use Technology**
   **a)** Use *The Geometer’s Sketchpad®* to draw your design from question 6.
   **b)** Use the measurement feature of *The Geometer’s Sketchpad®* to measure the area of your design.

8. **Chapter Problem** One of the gardens Emily is designing is made up of two congruent parallelograms.
   **a)** A plant is to be placed every 20 cm around the perimeter of the garden. Determine the number of plants Emily needs.
   **b)** Calculate the area of her garden.

9. **Use Technology** Use *The Geometer’s Sketchpad®* to create a composite figure made up of at least three different shapes.
   **a)** Estimate the perimeter and area of the figure you created.
   **b)** Determine the area using the measurement feature of *The Geometer’s Sketchpad®*. Was your estimate reasonable?
10. An archery target has a diameter of 80 cm. It contains a circle in the centre with a radius of 8 cm and four additional concentric rings each 8 cm wide.
   a) Find the area of the outer ring, to the nearest square centimetre.
   b) What percent of the total area is the outer ring?

11. The area of a square patio is 5 m².
   a) Find the length of one of its sides, to the nearest tenth of a metre.
   b) Find the perimeter of the patio, to the nearest metre.

12. Brandon works as a carpenter. He is framing a rectangular window that measures 1.5 m by 1 m. The frame is 10 cm wide and is made up of four trapezoids. Find the total area of the frame, to the nearest square centimetre.

### Achievement Check

13. Susan is replacing the shingles on her roof. The roof is made up of a horizontal rectangle on top and steeply sloping trapezoids on each side. Each trapezoid has a (slant) height of 4.5 m. The dimensions of the roof are shown in the top view.

   a) Calculate the area of the roof.
   b) A package of shingles covers 10 m². How many packages will Susan need to shingle the entire roof?
   c) Describe an appropriate way to round the number of packages in part b).
14. Sanjay is designing a square lawn to fit inside a square yard with side length 10 m so that there is a triangular flower bed at each corner.
   a) Find the area of Sanjay’s lawn.
   b) How does the area of the lawn compare to the area of the flower beds?
   c) Sanjay’s design is an example of a square *inscribed* within a square. The vertices of the inside square touch the sides of the outside square but do not intersect. Will your answer in part b) always be true when a square is inscribed within a square? Explain.

15. How does doubling the radius of a circle affect its area? Justify your answer using algebra.

16. Leonardo of Pisa lived in the 13th century in Pisa, Italy. He was given the nickname Fibonacci because his father’s name was Bonacci. Among his mathematical explorations is the sequence of numbers 1, 1, 2, 3, 5, 8, 13, 21, ….
   a) Determine the pattern rule for this sequence, and list the next four terms.
   b) Construct rectangles using consecutive terms for the sides. The first rectangle is 1 by 1, the second is 1 by 2, the third is 2 by 3, and so on. Find the area of each rectangle.
   c) Explore the ratios of the sides of the rectangles. Make conjectures about this ratio.
   d) Explore the ratios of the areas of the rectangles. Make conjectures about this ratio.

17. **Math Contest** Determine the ratio of the perimeter of the smallest square to the perimeter of the largest square.

18. **Math Contest** The midpoints of the sides of a rectangle that measures 10 cm by 8 cm are joined. Determine the area of the shaded region.
Investigate

How can you model the surface area and volume of a pyramid?

A: Surface Area

1. A square-based **pyramid** and its net are shown.
   
   **a)** What is the shape of the base? Write a formula for its area.
   
   **b)** What is the shape of each **lateral face**? Write a formula for the area of one lateral face.
   
   **c)** Write an expression for the surface area of the pyramid. Simplify the expression to give a formula for the surface area of a pyramid.

2. **Reflect** How would the results in step 1 change if you were developing a formula for the surface area of a hexagon-based pyramid? an octagon-based pyramid? Describe how to find the surface area of any pyramid.
Example 1  Surface Area of a Pyramid

A modern example of a pyramid can be found at the Louvre in Paris, France. The glass square-based pyramid was built as an entrance to this famous museum. Calculate the surface area of the pyramid, including the base area.

Solution

The surface area consists of the square base and the four congruent triangular faces.

\[
SA_{\text{pyramid}} = A_{\text{base}} + 4A_{\text{triangle}}
\]

\[
= (35)(35) + 4 \left[ \frac{1}{2} (35)(27.8) \right]
\]

\[
= 1225 + 1946
\]

\[
= 3171
\]

The surface area of the pyramid is 3171 m\(^2\).

B: Volume

1. a) Cut the top off the milk carton to form a prism.
   
   b) On a piece of construction paper, draw a net for a pyramid that has the same base and height as the prism.
   
   c) Cut out the net and tape it together to form a pyramid.

2. Estimate the ratio of the volume of the prism to the volume of the pyramid.

3. a) Cut along three sides of the base of the pyramid.
   
   Fill the pyramid with sand, rice, or another suitable material.
   
   b) Pour from the pyramid into the prism. How many pyramids full of material does it take to fill the prism?
   
   c) What fraction of the volume of the prism is the volume of the pyramid?

4. Reflect What conclusion can you draw about the relationship between the volume of a pyramid and the volume of a prism with the same base and height?

Optional

BLM 8.3.1 Net for a Pyramid

BLM 8.3.2 Net for a Prism

Tools

- an empty 250-mL milk carton
- construction paper
- scissors
- tape
- sand, rice, or another suitable material

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Example 2  Volume of a Pyramid

a) Determine the volume of the pyramid-shaped container, to the nearest cubic centimetre.

b) Express the capacity, in litres.

Solution

a) The volume of any pyramid can be determined using the formula

\[ V = \frac{1}{3} \text{(area of the base)(height)} \]

First, determine the height of the pyramid using the Pythagorean theorem.

\[ h^2 + 4^2 = 10^2 \]
\[ h^2 + 16 = 100 \]
\[ h^2 = 84 \]
\[ h = \sqrt{84} \]
\[ h \approx 9.2 \]

Now, calculate the volume.

\[ V = \frac{1}{3} \times 8^2 \times 9.2 \]
\[ = \frac{1}{3} \times 64 \times 9.2 \]
\[ \approx 196 \]

The volume of the container is about 196 cm\(^3\).

b) The capacity is the maximum volume that a container will hold. When a product is packaged, the container is usually not filled to capacity. This may be a factor to consider in some problems.

1 cm\(^3\) = 1 mL

The capacity of the container is 196 mL or 0.196 L of liquid.
Example 3 Surface Area and Volume of a Triangular Prism

Chocolate is sometimes packaged in a box that is shaped like a triangular prism.

a) Calculate the amount of material required to make this box, to the nearest square centimetre.

b) Calculate the volume of this box, to the nearest cubic centimetre.

Solution

a) The amount of material required is the surface area of the prism. The surface area consists of the top and bottom of the box, which are triangles, and the three faces, which are congruent rectangles.

First, determine the height, $h$, of each triangle.

Use the Pythagorean theorem.

$h^2 + 1.5^2 = 3^2$
$h^2 + 2.25 = 9$
$h^2 = 6.75$
$h = \sqrt{6.75}$
$h \approx 2.6$

Now, calculate the surface area.

$SA = 2A_{\text{base}} + 3A_{\text{face}}$

$= 2 \left[ \frac{1}{2} (3)(2.6) \right] + 3(3)(12)$

$= 7.8 + 108$

$= 115.8$

Approximately 116 cm$^2$ of material is needed to make the triangular prism box.

b) The volume of any right prism can be found using the formula $V = (\text{area of the base})(\text{height})$.

$V_{\text{triangular prism}} = (\text{area of the base})(\text{height})$

$= (3.9)(12)$

$= 46.8$

The volume is about 47 cm$^3$. 

Literacy Connections

A right prism has two parallel and congruent bases and lateral faces that are perpendicular to its bases. The triangular prism in Example 3 is a right triangular prism.
Key Concepts

- Surface area is a measure of how much material is required to cover or construct a three-dimensional object. Surface area is expressed in square units.

- The surface area of a prism or pyramid is the sum of the areas of the faces.

- Volume is a measure of how much space a three-dimensional object occupies. Capacity is the maximum volume a container can hold. Volume and capacity are measured in cubic units.

- The litre (L) is a measure of capacity or volume often used for liquids. 
  \[ 1 \text{ L} = 1000 \text{ cm}^3 \text{ or } 1 \text{ mL} = 1 \text{ cm}^3. \]

- For a prism, Volume = (area of base)(height).

\[ \text{height} \]
\[ \text{base} \]

- For a pyramid, Volume = \( \frac{1}{3} \) (area of base)(height).

\[ \text{height} \]
\[ \text{base} \]

Communicate Your Understanding

Describe how the shapes are alike. How are they different?

Describe how you would determine the volume of each shape.

Without doing the calculations, predict which shape has the greatest volume. Explain.

Describe how you would determine the surface area of each shape.

What unknown values would you need to find to complete the surface area calculations? Explain how you can determine the unknown values.
Practise

For help with question 1, see Example 1.

1. Determine the surface area of each object.
   a) ![Diagram of a triangular prism]
   b) ![Diagram of a square pyramid]

For help with question 2, see Example 2.

2. Determine the volume of each object. Round to the nearest cubic unit, when necessary.
   a) ![Diagram of a triangular prism]
   b) ![Diagram of a square pyramid]

For help with questions 3 to 5, see Example 3.

3. Determine the surface area of each object.
   a) ![Diagram of a rectangular prism]
   b) ![Diagram of a hexagonal pyramid]

4. Determine the volume of each object.
   a) ![Diagram of a rectangular prism]
   b) ![Diagram of a triangular prism]

5. A rectangular prism has length 3 m, width 2 m, and height 4 m.
   a) Determine the surface area of the prism.
   b) Determine the volume of the prism.

Connect and Apply

6. A cereal box has a volume of 3000 cm³. If its length is 20 cm and its width is 5 cm, what is its height?
7. Sneferu’s North Pyramid at Dahshur, Egypt, is shown. Its square base has side length 220 m and its height is 105 m.
   a) Determine the volume of this famous pyramid.
   b) Determine its surface area, to the nearest square metre.

8. The Pyramid of Khafre at Giza, Egypt, was built by the Pharaoh Khafre, who ruled Egypt for 26 years. The square base of this pyramid has side length 215 m and its volume is 2,211,096 m³. Calculate its height, to the nearest tenth of a metre.

9. The milk pitcher shown is a right prism. The base has an area of 40 cm² and the height of the pitcher is 26 cm. Will the pitcher hold 1 L of milk?

10. A juice container is a right prism with a base area of 100 cm².
    a) If the container can hold 3 L of juice, what is its height?
    b) Describe any assumptions you have made.

11. Adam has built a garden shed in the shape shown.
    a) Calculate the volume of the shed, to the nearest cubic metre.
    b) Adam plans to paint the outside of the shed, including the roof but not the floor. One can of paint covers 4 m². How many cans of paint will Adam need?
    c) If one can of paint costs $16.95, what is the total cost, including 7% GST and 8% PST?

12. **Chapter Problem** The diagram shows the side view of the swimming pool in Emily’s customer’s yard. The pool is 4 m wide.

    a) Estimate how many litres of water the pool can hold.
    b) Calculate how many litres of water the pool can hold.
    c) When the pool construction is complete, Emily orders water to fill it up. The water tanker can fill the pool at a rate of 100 L/min. How long will it take to fill the pool at this rate?
13. A triangular prism has a base that is a right triangle with shorter sides that measure 6 cm and 8 cm. The height of the prism is 10 cm.
   a) Predict how doubling the height affects the volume of the prism.
   b) Check your prediction by calculating the volume of the original prism and the volume of the new prism.
   c) Was your prediction accurate?
   d) Is this true in general? If so, summarize the result.

Achievement Check

14. a) Design two different containers that hold 8000 cm$^3$ of rice. One should be a rectangular prism and one should be a cylinder.
   b) Determine the surface area of each one, to the nearest square centimetre.
   c) Which shape would you recommend to the manufacturer and why?

Extend

15. A pyramid and a prism have congruent square bases. If their volumes are the same, how do their heights compare? Explain.

16. A statue is to be placed on a frustum of a pyramid. The frustum is the part remaining after the top portion has been removed by making a cut parallel to the base of the pyramid.
   a) Determine the surface area of the frustum.
   b) Calculate the cost of painting the frustum with gilt paint that costs $49.50/m$^2$. It is not necessary to paint the bottom of the frustum.

17. A formula for the surface area of a rectangular prism is $SA = 2(lw + lh + wh)$.
   a) Suppose each of the dimensions is doubled. Show algebraically how the surface area is affected.
   b) How is the volume affected if each of the dimensions is doubled? Justify your answer algebraically.

18. Math Contest A large wooden cube is made by glueing together 216 small cubes. Cuts are made right through the large cube along the diagonals of three perpendicular faces. How many of the small cubes remain uncut?
Investigate

How can you model the lateral area of a cone?

Work with a partner.

1. Construct a circle with a radius of 10 cm.

2. Draw two perpendicular radii and cut out and set the smaller sector of the circle aside to use later. What fraction of the circle is the larger piece?

3. Tape the radius edges on the large piece to form a cone. Measure the height, \( h \), of the cone and record it. Measure the radius, \( r \), of the base and record it.

4. Notice that \( h \) and \( r \) are sides in a right triangle. Calculate the length of the third side, \( s \). How is the length of the third side related to the circle you started with?

5. Calculate the circumference of the base of your cone. What fraction of the circumference of the original paper circle is this?
6. The curved surface of the cone is called the *lateral area*. What fraction of the area of your original paper circle is the lateral area of the cone?

7. **a)** Draw and cut out another circle with radius 10 cm. Draw any diameter and cut along the diameter. Construct a cone using the semicircle for the lateral area.

   **b)** Repeat steps 3 to 6 for this cone.

8. **a)** Use the smaller sector of the circle you cut out in question 2 to form another cone.

   **b)** Repeat steps 3 to 6 for this cone.

9. **Reflect** Describe the relationship between the fraction of the circumferences and the fraction of the areas.

You can use proportional reasoning to find the lateral area of a cone. The ratio of the areas is the same as the ratio of the circumferences.

\[
\frac{\text{Lateral area of cone}}{\text{Area of circle}} = \frac{\text{Circumference of cone}}{\text{Circumference of circle}}
\]

Consider a cone with slant height \(s\) and base radius \(r\).

The circumference of the large circle is \(2\pi s\) and the circumference of the base of the cone is \(2\pi r\).

The area of the large circle is \(\pi s^2\).

Substitute into the proportion:

\[
\frac{\text{Lateral area of cone}}{\pi s^2} = \frac{\text{Circumference of cone}}{2\pi s} = \frac{2\pi r}{2\pi s}
\]

\[
\frac{\text{Lateral area of cone}}{\pi s^2} = \frac{r}{s}
\]

Lateral area of cone = \(\frac{r}{s} \times \pi s^2\)

The lateral area of a cone with radius \(r\) and slant height \(s\) is \(\pi rs\).

The base of a cone is a circle with radius \(r\), so its area is \(\pi r^2\).

The total surface area of a cone is the sum of the areas of the base and the lateral surface.

\[
SA_{\text{cone}} = \pi rs + \pi r^2
\]
Example  Surface Area of a Cone

Calculate the surface area of the cone, to the nearest square centimetre.

Solution

To use the formula for the surface area of a cone, determine the slant height, s.

Use the Pythagorean theorem.

\[ s^2 = h^2 + r^2 \]
\[ s^2 = 8^2 + 3^2 \]
\[ s^2 = 64 + 9 \]
\[ s^2 = 73 \]
\[ s = \sqrt{73} \]
\[ s \approx 8.5 \]

The slant height of the cone is about 8.5 cm.

Now, use the formula for the surface area of a cone.

\[ SA_{\text{cone}} = \pi rs + \pi r^2 \]
\[ = \pi (3)(8.5) + \pi (3)^2 \]
\[ = 108 \]
\[ \approx 108 \text{ cm}^2 \]

The surface area of the cone is approximately 108 cm².

Key Concepts

- The surface area of a cone consists of the lateral area and the area of the circular base.
- The lateral area is formed by folding a sector of a circle. The radius of the circle used becomes the slant height, s, of the cone formed. The area of this curved surface is \( \pi rs \), where \( r \) is the radius of the base of the cone.
- The area of the circular base is \( \pi r^2 \).
- The formula for the surface area of a cone is \( SA_{\text{cone}} = \pi rs + \pi r^2 \).
- When you know the radius, \( r \), and height, \( h \), of a cone, you can determine the slant height, \( s \), using the Pythagorean theorem.
Communicate Your Understanding

C1 A cone is formed from a circle with a 90° sector removed. Another cone is formed from a semicircle with the same radius. How do the two cones differ? How are they the same?

C2 A cone is formed from a circle of radius 10 cm with a 60° sector removed. Another cone is formed from a circle of radius 15 cm with a 60° sector removed. How do the two cones differ? How are they the same?

C3 The slant height of a cone is doubled. Does this double the surface area of the cone? Explain your reasoning.

Practise

For help with questions 1 and 2, see the Example.

1. Calculate the surface area of each cone. Round to the nearest square unit.
   a) b) c)

2. a) Find the slant height of the cone.
   b) Calculate the surface area of the cone. Round to the nearest square metre.

Connect and Apply

3. Some paper cups are shaped like cones.
   a) How much paper, to the nearest square centimetre, is needed to make the cup?
   b) What assumptions have you made?
4. One cone has base radius 4 cm and height 6 cm. Another cone has a base radius 6 cm and height 4 cm.
   a) Do the cones have the same slant height?
   b) Do the cones have the same surface area? If not, predict which cone has the greater surface area. Explain your reasoning.
   c) Determine the surface area of each cone to check your prediction. Were you correct?

5. The lateral area of a cone with radius 4 cm is 60 cm².
   a) Determine the slant height of the cone, to the nearest centimetre.
   b) Determine the height of the cone, to the nearest centimetre.

6. The height of a cone is doubled. Does this double the surface area? Justify your answer.

7. The radius of a cone is doubled. Does this double the surface area? Justify your answer.

8. A cube-shaped box has sides 10 cm in length.
   a) What are the dimensions of the largest cone that fits inside this box?
   b) What is the surface area of this cone, to the nearest square centimetre?

9. A cone just fits inside a cylinder. The volume of the cylinder is 9425 cm³. What is the surface area of this cone, to the nearest square centimetre?

10. The frustum of a cone is the part that remains after the top portion has been removed by making a cut parallel to the base. Calculate the surface area of this frustum, to the nearest square metre.
11. **Chapter Problem** Emily has obtained an unfinished ceramic birdbath for one of her customers. She plans to paint it with a special glaze so that it will be weatherproof. The birdbath is constructed of two parts:
- a shallow open-topped cylinder with an outside diameter of 1 m and a depth of 5 cm, with 1-cm-thick walls and base
- a conical frustum on which the cylinder sits

![Diagram of birdbath](image)

a) Identify the surfaces that are to be painted and describe how to calculate the area.

b) Calculate the surface area to be painted, to the nearest square centimetre.

c) One can of glaze covers 1 m\(^2\). How many cans of glaze will Emily need to cover all surfaces of the birdbath and the frustum?

12. Create a problem involving the surface area of a cone. Solve the problem. Exchange with a classmate.

**Extend**

13. Suppose the cube in question 8 has sides of length \(x\).
   a) Write expressions for the dimensions of the largest cone that fits inside this box.
   b) What is a formula for the surface area of this cone?

14. a) Find an expression for the slant height of a cone in terms of its lateral area and its radius.
   b) If the lateral area of a cone is 100 cm\(^2\) and its radius is 4 cm, determine its slant height.

15. Located in the Azores Islands off the coast of Portugal, Mt. Pico Volcano stands 2351 m tall. Measure the photo to estimate the radius of the base of the volcano, and then calculate its lateral surface area, to the nearest square metre.

### Did You Know?

There are 8000 to 10 000 people of Azorean heritage living in Ontario.
16. **Use Technology** A cone has a radius of 2 cm.

a) Write an algebraic model for the surface area of this cone in terms of its slant height.

b) Use *The Geometer's Sketchpad®* to investigate how the surface area of a cone changes as the slant height changes. Since *The Geometer's Sketchpad®* cannot easily show three-dimensional objects, represent the cone with a triangle that is a side view of the cone.

• From the **Edit** menu, choose **Preferences**. Click on the **Text** tab. Ensure that **For All New Points** is checked.

• Draw a point A. Select point A. From the **Transform** menu, choose **Translate**. Ensure that the **Polar**, **Fixed Distance**, and **Fixed Angle** radio buttons are on. Change the distance to 2 cm and the angle to 0°. Click on **Translate**. Point A’ will appear 2 cm to the right of point A. Draw another point 2 cm to the left of point A, using an angle of 180°.

• Construct a line segment joining the three points. Select point A and the line segment. From the **Construct** menu, choose **Perpendicular Line** to draw a perpendicular line through point A.

• Draw a point B on the line above point A. Construct line segments to form a triangle. This triangle represents the side view of a cone with a variable height AB and a fixed radius of 2 cm.

• Measure the radius of the cone. Select this measurement. Right click and choose **Label Measurement** from the drop-down menu. Change the label to *r*.

• Measure the slant height of the cone. Change the label to *s*.

• Select *r* and *s*. From the **Measure** menu, choose **Calculate**. Enter the formula \( \pi r^2 + \pi rs \) by selecting \( \pi \), *r*, and *s* from the **Values** drop-down menu on the calculator. Change the label to **SA**. This is the surface area of the cone. Drag point B back and forth along the line. Watch how the measurements change.

• Select *s* and then **SA**. From the **Graph** menu, choose **Tabulate**. Move the table to a convenient location. Move point B, and note how the values in the table change.

• Adjust the value of *s* to about 3 cm. Select the table. From the **Graph** menu, choose **Add Table Data**. Click on **OK**. Repeat this process with *s* set to about 4 cm. Continue until you have five sets of data.

From the **Graph** menu, choose **Plot Table Data**. You will see a graph of the data that you have collected.

c) Describe the relationship that resulted from this investigation using mathematical terms.
Volume of a Cone

Cone-shaped containers are used in a variety of professions, such as environmental studies, agriculture, and culinary arts.

In this section, you will develop a formula for the volume of any cone.

Investigate

How can you model the volume of a cone?

Work with a partner.

1. Measure the radius and height of the can.

2. Construct a cone with the same base radius and height as the can.
   a) Use the radius and height to calculate the slant height of the cone.
   b) Construct a circle with a radius equal to the slant height you determined. Make a cut along a radius so that the circle can be formed into a cone.
   c) The cone’s circumference should fit the circumference of the can. Tape the seam to form a cone.

3. Fill the cone with sand, rice, or another suitable material. Empty the rice into the can. Repeat until the can is full. How many cones of material does it take to fill the can?

4. a) Reflect What conclusion can you draw about the relationship between the volume of a cone and the volume of a cylinder with the same height and radius?
   b) You know the formula for the volume of a cylinder. Use your conclusion from part a) to write a formula for the volume, \( V \), of a cone in terms of the radius, \( r \), of the base and the height, \( h \).
Example 1  Volume of a Frozen Yogurt Treat

Tracy makes her own frozen yogurt treats in cone-shaped paper cups. Determine the volume of the frozen yogurt treat shown, to the nearest cubic centimetre.

Solution

The volume of a cone is one third the volume of the cylinder with the same base and height.

\[ V_{\text{cone}} = \frac{1}{3} \text{(volume of a cylinder)} \]
\[ = \frac{1}{3} \pi r^2 h \]
\[ = \frac{1}{3} \pi (4^2)(12) \quad \text{Estimate:} \frac{1}{3}(3)(4^2)(12) = (16)(12) \]
\[ \approx 201 \quad = 192 \]

The volume of the frozen yogurt treat is approximately 201 cm\(^3\).

Example 2  Volume of a Sand Pile

A conical pile of sand has a base diameter of 10 m and a slant height of 8 m. Determine the volume of the sand in the pile, to the nearest cubic metre.

Solution

Since the diameter of the base is 10 m, the radius is 5 m.

To determine the volume of the cone, you need to know the height.

Apply the Pythagorean theorem.

\[ s^2 = h^2 + r^2 \]
\[ 8^2 = h^2 + 5^2 \]
\[ 64 = h^2 + 25 \]
\[ h^2 = 64 - 25 \]
\[ h^2 = 39 \]
\[ h = \sqrt{39} \]
\[ h \approx 6.2 \]

The height of the cone is approximately 6.2 m.
Now, determine the volume.

\[ V_{\text{cone}} = \frac{1}{3} \pi r^2 h \]

\[ = \frac{1}{3} \pi (5)^2 (6.2) \quad \text{Estimate: } \frac{1}{3} (3)(5^2)(6) = (25)(6) = 150 \]

\[ \div 162 \quad 1 \times 3 \times \pi \times 5 \div 6.2 = 162.3156204 \]  

The volume of the sand in the pile is approximately 162 m³.

**Example 3  Find the Height of a Container**

A fountain firework is packaged in a conical container. Its volume is 210 cm³. Its diameter is 8 cm. What is the height of the fountain firework, to the nearest tenth of a centimetre?

**Solution**

Substitute the given values into the formula for the volume of a cone.

\[ V_{\text{cone}} = \frac{1}{3} \pi r^2 h \]

\[ 210 = \frac{1}{3} \pi (4)^2 h \]

\[ 210 = \frac{16\pi}{3} h \]

\[ 210 \times \frac{3}{16\pi} = h \]

\[ h \approx 12.5 \]

The height of the conical firework is approximately 12.5 cm.

**Key Concepts**

- The volume of a cone is one third the volume of a cylinder with the same base radius and height:
  \[ V_{\text{cone}} = \frac{1}{3} \pi r^2 h \]

- If you know the slant height, \( s \), and base radius, \( r \), of a cone, you can use the Pythagorean theorem to determine the height, \( h \), of the cone.
Communicate Your Understanding

**C1** A cylindrical container and a conical container have the same radius and height. How are their volumes related? How could you illustrate this relationship for a friend?

**C2** Suppose the height of a cone is doubled. How will this affect the volume?

**C3** Suppose the radius of a cone is doubled. How will this affect the volume?

Practise

For help with question 1, see Example 1.

1. Determine the volume of each cone. Round to the nearest cubic unit.

   ![Diagram of cone a) with dimensions: radius = 6 cm, height = 2 cm]

   ![Diagram of cone b) with dimensions: radius = 6.4 m, height = 5.3 m]

   ![Diagram of cone c) with dimensions: radius = 12 mm, height = 30 mm]

   ![Diagram of cone d) with dimensions: radius = 60 cm, height = 40 cm]

For help with questions 2 and 3, see Example 2.

2. Determine the volume of each cone. Round to the nearest cubic unit.

   ![Diagram of cone a) with dimensions: radius = 2 m, height = 1 m]

   ![Diagram of cone b) with dimensions: radius = 30 cm, height = 10 cm]

3. Wesley uses a cone-shaped funnel to put oil in a car engine. The funnel has a radius of 5.4 cm and a slant height of 10.2 cm. How much oil can the funnel hold, to the nearest tenth of a cubic centimetre?
For help with question 4, see Example 3.

4. A cone-shaped paper cup has a volume of 67 cm³ and a diameter of 6 cm. What is the height of the paper cup, to the nearest tenth of a centimetre?

Connect and Apply

5. A cone just fits inside a cylinder with volume 300 cm³. What is the volume of the cone?


7. A cone has a volume of 150 cm³. What is the volume of a cylinder that just holds the cone?

8. A cone-shaped storage unit at a highway maintenance depot holds 4000 m³ of sand. The unit has a base radius of 15 m.
   a) Estimate the height of the storage unit.
   b) Calculate the height.
   c) How close was your estimate?

9. A cone has a height of 4 cm and a base radius of 3 cm. Another cone has a height of 3 cm and a base radius of 4 cm.
   a) Predict which cone has the greater volume. Explain your prediction.
   b) Calculate the volume of each cone, to the nearest cubic centimetre. Was your prediction correct?

10. Chapter Problem Refer to question 11 in Section 8.4. Determine the volume of concrete in Emily’s birdbath. Round your answer to the nearest cubic centimetre.

11. a) Express the height of a cone in terms of its volume and its radius.
    b) If a cone holds 1 L and its radius is 4 cm, what is its height? Round your answer to the nearest tenth of a centimetre.

12. A cone-shaped funnel holds 120 mL of water. If the height of the funnel is 15 cm, determine the radius, rounded to the nearest tenth of a centimetre.
Extend

13. A cone just fits inside a cube with sides that measure 10 cm.
   a) What are the dimensions of the largest cone that fits inside this box?
   b) Estimate the ratio of the volume of the cone to the volume of the cube.
   c) Calculate the volume of the cone, to the nearest cubic centimetre.
   d) Calculate the ratio in part b).
   e) How close was your estimate?

14. A cone has a height equal to its diameter. If the volume of the cone is 200 m³, determine the height of the cone, to the nearest tenth of a metre.

15. Use Technology Use a graphing calculator, The Geometer’s Sketchpad®, or a spreadsheet to investigate how the volume of a cone is affected when its radius is constant and its height changes.

16. Use Technology A cone has a height of 20 cm.
   a) Write an algebraic model for the volume of the cone in terms of the radius.
   b) Choose a tool for graphing. Graph the volume of the cone versus the radius.
   c) Describe the relationship using mathematical terms.

17. Math Contest A cube has side length 6 cm. A square-based pyramid has side length 6 cm and height 12 cm. A cone has diameter 6 cm and height 12 cm. A cylinder has diameter 6 cm and height 6 cm. Order the figures from the least to the greatest volume. Select the correct order.
   A cube, pyramid, cone, cylinder
   B cylinder, cube, cone, pyramid
   C cube, cone, cylinder, pyramid
   D cone, pyramid, cylinder, cube
   E pyramid, cone, cylinder, cube
A **sphere** is a three-dimensional shape that is often seen in sports. Balls of different sizes are used to play basketball, soccer, and volleyball, to name a few.

Consider the balls shown. What shapes appear to make up the surface of each sphere? How could you use the area of these shapes to help you find the surface area of a sphere?

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**Investigate**

How can you model the surface area of a sphere?

Work with a partner.

1. Choose an orange that is as spherical as possible. Estimate the surface area of the orange, in square centimetres.

2. **a)** Measure the circumference of the orange. Use a piece of string to go around the outside of the orange. Then, measure the length of the string.  
   **b)** Use the formula for the circumference of a circle, \( C = 2\pi r \), to find the radius of the orange.

3. Carefully peel the orange. Flatten the pieces and place them on grid paper. Trace around the pieces. Find the area of the peel by counting squares and partial squares on the grid paper.

4. **a)** Determine the area of a circle with the same radius as the orange.  
   **b)** What is the approximate ratio of the orange’s surface area to the area of the circle?  
   **c)** Describe a possible formula for the surface area of a sphere based on your results.

5. **Reflect** Compare your results with those of your classmates. What do you conclude that the formula for the surface area of a sphere is?
Example 1  Surface Area of an Eyeball

The dimensions of an adult human eyeball are reasonably constant, varying only by a millimetre or two. The average diameter is about 2.5 cm. Calculate the surface area of the human eyeball, to the nearest tenth of a square centimetre.

Solution

The formula for the surface area of a sphere is $SA_{\text{sphere}} = 4\pi r^2$.

The eyeball has a diameter of 2.5 cm, so the radius is 1.25 cm.

$SA_{\text{sphere}} = 4\pi r^2$

$= 4\pi(1.25)^2$  \[\text{Estimate: } 4(3)(1)^2 = 12\]

$\approx 19.6$

The surface area of the human eyeball is about 19.6 cm$^2$.

Example 2  Find the Radius of a Baseball

Determine the radius of a baseball that has a surface area of 215 cm$^2$. Round your answer to the nearest tenth of a centimetre.

Solution

Substitute the values into the formula.

$SA_{\text{sphere}} = 4\pi r^2$

$\frac{215}{4\pi} = r^2$

$\sqrt{\frac{215}{4\pi}} = r$

$r \approx 4.1$

The radius of the baseball is about 4.1 cm.

Key Concepts

- The formula for the surface area of a sphere with radius $r$ is $SA_{\text{sphere}} = 4\pi r^2$.
- If you know the surface area of a sphere, you can determine the radius, $r$, of the sphere.
Communicate Your Understanding

C1 Describe how you would determine the amount of leather required to cover a softball.

C2 Does doubling the radius of a sphere double the surface area? Explain your reasoning.

Practise

For help with questions 1 and 2, see Example 1.

1. Determine the surface area of each sphere. Round to the nearest square unit.
   
   a) ![Image of a sphere with a diameter of 6 cm]
   
   b) ![Image of a sphere with a diameter of 30.2 mm]
   
   c) ![Image of a sphere with a diameter of 3 m]
   
   d) ![Image of a sphere with a diameter of 5.6 m]

2. A ball used to play table tennis has a diameter of 40 mm.
   
   a) Estimate the surface area of this ball.
   
   b) Calculate the surface area, to the nearest square millimetre. How close was your estimate?

For help with question 3, see Example 2.

3. A sphere has a surface area of 42.5 m². Find its radius, to the nearest tenth of a metre.

Connect and Apply

4. A basketball has a diameter of 24.8 cm.
   
   a) How much leather is required to cover this ball, to the nearest tenth of a square centimetre?
   
   b) If the leather costs $28/m², what does it cost to cover the basketball?

5. The diameter of Earth is approximately 12 800 km.
   
   a) Calculate the surface area of Earth, to the nearest square kilometre.
   
   b) What assumptions did you make?
6. a) The diameter of Mars is 6800 km. Calculate its surface area, to the nearest square kilometre.
   b) Compare the surface area of Mars to the surface area of Earth from question 5. Approximately how many times greater is the surface area of Earth than the surface area of Mars?

7. **Chapter Problem** Emily is placing a gazing ball in one of her customer’s gardens. The ball has a diameter of 60 cm and will be covered with reflective crystals. One jar of these crystals covers 1 m².
   a) Estimate the surface area to decide whether one jar of the crystals will cover the ball.
   b) Calculate the surface area, to the nearest square centimetre.
   c) Was your estimate reasonable? Explain.

8. The radius of a sphere is 15 cm.
   a) Predict how much the surface area increases if the radius increases by 2 cm.
   b) Calculate the change in the surface area, to the nearest square centimetre.
   c) How accurate was your prediction?

9. **Use Technology**
   a) Use a graphing calculator to graph the surface area of a sphere versus its radius by entering the surface area formula.
   b) Describe the relationship.
   c) Use the TRACE feature to determine
      • the surface area of a sphere with radius 5.35 cm
      • the radius of a sphere with surface area 80 cm²

**Extend**

10. **Use Technology**
    a) Determine an algebraic expression for the radius of a sphere in terms of its surface area.
    b) Use your expression from part a) and a graphing calculator to graph the relationship between the radius and the surface area.
    c) Describe the relationship.
    d) Use the graphing calculator to find the radius of a sphere with surface area 200 cm².

11. A spherical balloon is blown up from a diameter of 10 cm to a diameter of 30 cm. By what factor has its surface area increased? Explain your reasoning.

12. Which has the larger surface area: a sphere of radius \( r \) or a cube with edges of length \( 2r \)?
13. **Use Technology** A sphere just fits inside a cube with sides of length 10 cm.

a) Estimate the ratio of the surface area of the sphere to the surface area of the cube.

b) Calculate the surface areas of the sphere and the cube and their ratio.

c) How does your answer compare to your estimate?

d) Use *The Geometer’s Sketchpad®* to investigate this relationship for any size of cube with an inscribed sphere. Since *The Geometer’s Sketchpad®* cannot easily show three-dimensional objects, represent the cube with a square and the sphere with a circle.

- From the **Edit** menu, choose **Preferences**. Click on the **Text** tab. Ensure that **For All New Points** is checked.
- Select the **Custom Tool**. From the drop-down menu, choose **Polygons** and then **4/Square (By Edge)**. Draw a square ABCD.
- Construct the diagonals of the square. Draw a circle with its centre at E, where the diagonals cross, such that it is inscribed in the square. Draw a radius EF.
- Measure radius EF of the circle. Select this measurement. Right click and choose **Label Measurement** from the drop-down menu. Change the label to r.
- Measure side AB of the square. Change the label to s.
- Select s. From the **Measure** menu, choose **Calculate**. Enter the formula $6 \times s^2$ by selecting s from the **Values** drop-down menu on the calculator. Change the label to **SA of Cube**.
- Select r. From the **Measure** menu, choose **Calculate**. Enter the formula $4 \times \pi \times r^2$ by selecting r from the **Values** drop-down menu on the calculator. Change the label to **SA of sphere**.
- Calculate the ratio $\frac{SA \ of \ cube}{SA \ of \ sphere}$.

- Drag point A. Watch how the measurements change.

What can you conclude about the ratio of the surface areas of a cube and a sphere inscribed in the cube?
The annual Gatineau Hot Air Balloon Festival has been held since 1988 in Gatineau, Québec. Hot air balloons come in a variety of shapes. One special shape that appeared at the festival was a soccer ball. This soccer ball was about 17.4 m in diameter, and could hold 385 696 ordinary soccer balls inside it.

In this section, you will develop a formula for the volume of any sphere.

Investigate

How can you model the volume of a sphere?

Work in small groups.

1. Before you take any measurements, estimate the volume of one tennis ball.

2. a) Measure the diameter of the cylinder. It should be almost the same as the diameter of the tennis ball.
   
   b) Measure the height of the cylinder. It should be almost the same as three times the diameter of the tennis ball.

3. Place the cylinder in the overflow container and fill the cylinder with water.

4. Slowly place the three tennis balls inside the cylinder, one at a time, allowing the water to overflow into the container. Push the balls down to the bottom of the cylinder.

5. Remove the tennis balls from the cylinder, take the can out of the overflow container, and empty the water from the cylinder into the sink. Pour the water from the overflow container back into the cylinder. Measure the depth of the water.
Example 1 Volume of Pluto

Pluto is the smallest planet in the solar system. The diameter of Pluto is approximately 2290 km. Calculate the volume of Pluto.

Solution

The volume of a sphere is two thirds the volume of a cylinder with the same radius and a height equal to the diameter of the sphere. If the sphere has radius \( r \), then the cylinder has a base radius \( r \) and height \( 2r \).

\[
V_{\text{sphere}} = \frac{2}{3} (\text{volume of a cylinder})
\]

\[
= \frac{2}{3} \pi r^2 h
\]

\[
= \frac{2}{3} \pi (r^2)(2r)
\]

\[
= \frac{4}{3} \pi r^3
\]

The radius is one half the diameter. The radius of Pluto is 1145 km. Use the formula for the volume of a sphere.

\[
V_{\text{sphere}} = \frac{4}{3} \pi r^3
\]

\[
= \frac{4}{3} \pi (1145)^3
\]

\[
= 6300000000\text{ km}^3
\]

The volume of Pluto is approximately 6 300 000 000 km\(^3\).

6. What fraction of the cylinder is filled with water? How does the volume of this displaced water compare to the volume of the three tennis balls?

7. If the cylinder were only big enough to hold one tennis ball, what fraction of the can would be filled with water?

8. Reflect How does the volume of a sphere compare to the volume of a cylinder? How would you calculate the volume of one tennis ball?

9. Use your method to calculate the volume of one tennis ball. How does your answer compare to your estimate?
Example 2  Package a Gemstone

A spherical gemstone just fits inside a plastic cube with edges 10 cm.

a) Calculate the volume of the gemstone, to the nearest cubic centimetre.

b) How much empty space is there?

Solution

a) The diameter of the gemstone is about 10 cm, so its radius is 5 cm.

\[
V_{\text{sphere}} = \frac{4}{3} \pi r^3
\]

\[
= \frac{4}{3} \pi (5)^3 \quad \text{Estimate:} \quad \frac{4}{3} (3)^3 = 4(125) \quad \frac{524}{500}
\]

The volume of the gemstone is about 524 cm$^3$.

b) Determine the volume of the cube.

\[
V_{\text{cube}} = s^3
\]

\[
= 10^3
\]

\[
= 1000
\]

The empty space is the difference in the volumes of the cube and the gemstone.

\[
V_{\text{empty space}} = V_{\text{cube}} - V_{\text{sphere}}
\]

\[
= 1000 - 524
\]

\[
= 476
\]

There is about 476 cm$^3$ of empty space in the box.

Key Concepts

- The volume of a sphere with radius $r$ is given by the formula $V_{\text{sphere}} = \frac{4}{3} \pi r^3$.

- You can calculate the empty space in a container by subtracting the volume of the object from the volume of the container in which it is packaged.
Communicate Your Understanding

C1 Describe how you would determine the volume of a sphere if you knew its surface area.

C2 How is the volume of a sphere affected if you double the radius?

Practise

For help with questions 1 to 3, see Example 1.

1. Calculate the volume of each sphere. Round to the nearest cubic unit.
   a) [Image of sphere with diameter 14.2 cm]
   b) [Image of sphere with diameter 32 mm]
   c) [Image of sphere with diameter 2.1 m]

2. A golf ball has a diameter of 4.3 cm. Calculate its volume, to the nearest cubic centimetre.

3. Hailstones thought to be the size of baseballs killed hundreds of people and cattle in the Moradabad and Beheri districts of India in 1888. The hailstones had a reported diameter of 8 cm. What was the volume of each one, to the nearest cubic centimetre?

For help with question 4, see Example 2.

4. A table tennis ball just fits inside a plastic cube with edges 40 mm.
   a) Calculate the volume of the table tennis ball, to the nearest cubic millimetre.
   b) Calculate the volume of the cube.
   c) Determine the amount of empty space.
5. The largest lollipop ever made had a diameter of 140.3 cm and was made for a festival in Gränna, Sweden, on July 27, 2003.
   a) If a spherical lollipop with diameter 4 cm has a mass of 50 g, what was the mass of this giant lollipop to the nearest kilogram?
   b) Describe any assumptions you have made.

6. **Chapter Problem** Emily orders a spherical gazing ball for one of her customers. It is packaged tightly in a cylindrical container with a base radius of 30 cm and a height of 60 cm.
   a) Calculate the volume of the sphere, to the nearest cubic centimetre.
   b) Calculate the volume of the cylindrical container, to the nearest cubic centimetre.
   c) What is the ratio of the volume of the sphere to the volume of the container?
   d) Is this ratio consistent for any sphere that just fits inside a cylinder? Explain your reasoning.

7. Golf balls are stacked three high in a rectangular prism package. The diameter of one ball is 4.3 cm. What is the minimum amount of material needed to make the box?

8. A cylindrical silo has a hemispherical top (half of a sphere). The cylinder has a height of 20 m and a base diameter of 6.5 m.
   a) Estimate the total volume of the silo.
   b) Calculate the total volume, to the nearest cubic metre.
   c) The silo should be filled to no more than 80% capacity to allow for air circulation. How much grain can be put in the silo?
   d) A truck with a bin measuring 7 m by 3 m by 2.5 m delivers grain to the farm. How many truckloads would fill the silo to its recommended capacity?

9. The tank of a propane tank truck is in the shape of a cylinder with a hemisphere at both ends. The tank has a radius of 2 m and a total length of 10.2 m. Calculate the volume of the tank, to the nearest cubic metre.
10. Estimate how many basketballs would fit into your classroom. Explain your reasoning and estimation techniques and describe any assumptions you have made. Compare your answer with that of a classmate. Are your answers close? If not, whose answer is a more reasonable estimate and why?

Achievement Check

11. The T-Ball company is considering packaging two tennis balls that are 8.5 cm in diameter in a cylinder or in a square-based prism.
   a) What are the dimensions and volumes of the two containers?
   b) How much empty space would there be in each container?
   c) What factors should the T-Ball company consider in choosing the package design? Justify your choices.

Extend

12. Estimate and then calculate the radius of a sphere with a volume of 600 cm³.

13. Use Technology Graph \( V = \frac{4}{3} \pi r^3 \) using a graphing calculator.
   a) Use the TRACE feature to determine the volume of a sphere with a radius of 6.2 cm.
   b) Check your answer to question 12 by using the TRACE feature to approximate the radius of a sphere with a volume of 600 cm³.

14. If the surface area of a sphere is doubled from 250 cm² to 500 cm², by what factor does its volume increase?

15. A sphere just fits inside a cube with sides of length 8 cm.
   a) Estimate the ratio of the volume of the sphere to the volume of the cube.
   b) Calculate the volumes of the sphere and the cube and their ratio.
   c) How does your answer compare to your estimate?

16. Which has the larger volume: a sphere of radius \( r \) or a cube with edges of length \( 2r \)?
17. Use Technology

Use *The Geometer’s Sketchpad®* to investigate how the volume of glass required to make a spherical light bulb of constant thickness 0.2 cm changes as the radius of the light bulb changes. Since *The Geometer’s Sketchpad®* cannot easily show three-dimensional objects, represent the spherical light bulb with two concentric circles 0.2 cm apart.

- From the *Edit* menu, choose *Preferences*. Click on the *Text* tab. Ensure that *For All New Points* is checked.
- Draw a point A. Select point A. From the *Transform* menu, choose *Translate*. Ensure that the *Polar*, *Fixed Distance*, and *Fixed Angle* radio buttons are on. Change the distance to 1 cm and the angle to 0°. Click on *Translate*. Point A’ will appear 1 cm to the right of point A. Construct a line through points A and A’.
- Draw a circle with centre A and radius AB such that point B is to the right of A’, and on the line.
- With point B selected, choose *Translate* from the *Transform* menu. Ensure that the *Polar*, *Fixed Distance*, and *Fixed Angle* radio buttons are on. Change the distance to 0.2 cm and the angle to 0°. Click on *Translate*. Point B’ will appear 0.2 cm to the right of point B.
- Draw a circle with centre A and radius AB’.
- Measure the inner radius AB. Select this measurement. Right click and choose *Label Distance Measurement* from the drop-down menu. Change the label to *Inner r*.
- Measure the outer radius AB’. Change the label to *Outer r*.
- Select *Inner r*. From the *Measure* menu, choose *Calculate*. Enter the formula \(4/3 \times \pi \times \text{Inner } r^3\) by selecting *Inner r* from the *Values* drop-down menu on the calculator. Change the label to *Inner V*. This is the volume of the sphere inside the light bulb.
- Select *Outer r*. From the *Measure* menu, choose *Calculate*. Enter the formula \(4/3 \times \pi \times \text{Outer } r^3\) by selecting *Outer r* from the *Values* drop-down menu on the calculator. Change the label to *Outer V*. This is the outer volume of the light bulb.
• Calculate the value $\text{Outer V} - \text{Inner V}$. Change the label to Glass V. This is the volume of glass required to make the light bulb.

• Select Outer $r$ and then Glass V. From the Graph menu, choose Tabulate. Move the table to a convenient location. Move point B, and note how the values in the table change.

• Adjust the value of Outer $r$ to about 1 cm. Select the table. From the Graph menu, choose Add Table Data. Click on OK. Adjust Outer $r$ to about 2 cm. Choose Add Table Data again. Continue until you have five sets of data.

• From the Graph menu, choose Plot Table Data. You will see a graph of the data that you have collected.

Describe the relationship in mathematical terms.

18. Math Contest A cylinder has radius 6 cm and height 6 cm. A cone has radius 6 cm and height 6 cm. A sphere has radius 6 cm. Order the figures from least volume to greatest volume. Select the correct order.

A cone, sphere, cylinder
B cone, cylinder, sphere
C sphere, cone, cylinder
D cylinder, sphere, cone
E cylinder, cone, sphere

19. Math Contest A dozen of Terry’s favourite golf balls are sold in a rectangular box. Each ball has a diameter of 4 cm. Determine the volume of empty space in the box of golf balls.
8.1 Apply the Pythagorean Theorem, pages 418–425

1. Determine the perimeter and area of each right triangle. Round answers to the nearest tenth of a unit or square unit.

   a) \[ \triangle \text{with sides 10.5 cm and 8.2 cm} \]

   b) \[ \triangle \text{with sides 12 m and 6 m} \]

2. A 6-m extension ladder leans against a vertical wall with its base 2 m from the wall. How high up the wall does the top of the ladder reach? Round to the nearest tenth of a metre.

8.2 Perimeter and Area of Composite Figures, pages 426–435

3. Calculate the perimeter and area of each figure. Round answers to the nearest tenth of a unit or square unit, if necessary.

   a) \[ \text{figure with sides 3 m, 6 m, and 5 m} \]

   b) \[ \text{triangle with sides 8 cm and 10 cm} \]

4. The diagram shows a running track at a high school. The track consists of two parallel line segments, with a semicircle at each end. The track is 10 m wide.

   a) Tyler runs on the inner edge of the track. How far does he run in one lap, to the nearest tenth of a metre?

   b) Dylan runs on the outer edge. How far does he run in one lap, to the nearest tenth of a metre?

   c) Find the difference between the distances run by Tyler and Dylan.

8.3 Surface Area and Volume of Prisms and Pyramids, pages 436–443

5. Calculate the surface area of each object. Round answers to the nearest square unit, if necessary.

   a) \[ \text{cuboid with dimensions 10 cm, 5 cm, and 4 cm} \]

   b) the Great Pyramid of Cheops, with a height of about 147 m and a base width of about 230 m
6. a) Calculate the volume of the tent.

![Diagram of tent]

b) How much nylon is required to make this tent?
c) Describe any assumptions you made in part b).
d) How reasonable is your answer in part b)?

7. A cylindrical can holds 500 mL and has a radius of 4 cm. Calculate the height of the can, to the nearest tenth of a centimetre.

8.4 Surface Area of a Cone, pages 444–450

8. Calculate the surface area of a cone with a slant height of 13 cm and a height of 12 cm. Round to the nearest square centimetre.

9. The cone portion of a traffic pylon has a diameter of 20 cm and a vertical height of 35 cm. Calculate the surface area of the cone portion of the pylon, to the nearest square centimetre. Assume that the bottom of the cone is complete.

8.5 Volume of a Cone, pages 451–456

10. A conical funnel holds 100 mL. If the height of the funnel is 10 cm, determine its radius, to the nearest tenth of a centimetre.

11. Calculate the volume of a cone that just fits inside a cylinder with a base radius of 8 cm and a height of 10 cm. Round to the nearest cubic centimetre. How does the volume of the cone compare to the volume of the cylinder?

8.6 Surface Area of a Sphere, pages 457–461

12. A volleyball has a diameter of 21.8 cm. Calculate the amount of leather required to cover the volleyball, to the nearest tenth of a square centimetre.

13. The diameter of Earth is about 12 800 km.
   a) Calculate the area of the Northern Hemisphere, to the nearest square kilometre.
   b) What assumptions have you made?
   c) Canada’s area is 9 970 610 km². Estimate the fraction of the Northern Hemisphere that Canada covers.

8.7 Volume of a Sphere, pages 462–469

14. Calculate the volume of a soccer ball with a diameter of 22.3 cm, to the nearest tenth of a cubic centimetre.

15. The soccer ball in question 14 is packaged so that it just fits inside a cube-shaped box.
   a) Estimate the amount of empty space inside the box.
   b) Calculate the amount of empty space.
   c) How close was your estimate?
Multiple Choice

For questions 1 to 5, select the best answer.

1. A sphere has a radius of 3 cm. What is its volume, to the nearest cubic centimetre?
   A 339 cm³
   B 38 cm³
   C 113 cm³
   D 85 cm³

2. What is the area of the figure, to the nearest square centimetre?
   A 43 cm²
   B 54 cm²
   C 62 cm²
   D 73 cm²

3. A circular swimming pool has a diameter of 7.5 m. It is filled to a depth of 1.4 m. What is the volume of water in the pool, to the nearest litre?
   A 61 850 L
   B 247 400 L
   C 23 561 L
   D 47 124 L

4. A conical pile of road salt is 15 m high and has a base diameter of 30 m. How much plastic sheeting is required to cover the pile, to the nearest square metre?
   A 414 m²
   B 990 m²
   C 707 m²
   D 999 m²

5. What is the length of the unknown side of the triangle, to the nearest tenth of a millimetre?
   A 2.3 mm
   B 5.0 mm
   C 6.1 mm
   D 7.7 mm

Short Response

Show all steps to your solutions.

6. A candle is in the shape of a square-based pyramid.

a) How much wax is needed to create the candle, to the nearest cubic centimetre?

b) How much plastic wrap, to the nearest tenth of a square centimetre, would you need to completely cover the candle? What assumptions did you make?

7. A rectangular cardboard carton is designed to hold six rolls of paper towel that are 28 cm high and 10 cm in diameter. Describe how you would calculate the amount of cardboard required to make this carton.

8. Compare the effects of doubling the radius on the volume of a cylinder and a sphere. Justify your answer with numerical examples.

9. Calculate the surface area of the cone that just fits inside a cylinder with a base radius of 8 cm and a height of 10 cm. Round to the nearest square centimetre.

10. Determine the volume of a conical pile of grain that is 10 m high with a base diameter of 20 m. Round to the nearest cubic metre.
Extended Response

Provide complete solutions.

11. Three tennis balls that measure 8.4 cm in diameter are stacked in a cylindrical can.

a) Determine the minimum volume of the can, to the nearest tenth of a cubic centimetre.
b) Calculate the amount of aluminum required to make the can, including the top and bottom. Round to the nearest square centimetre.
c) The can comes with a plastic lid to be used once the can is opened. Find the amount of plastic required for the lid. Round to the nearest square centimetre.
d) Describe any assumptions you have made.

12. A rectangular carton holds 12 cylindrical cans that each contain three tennis balls, like the ones described in question 11.

a) How much empty space is in each can of tennis balls, to the nearest tenth of a cubic centimetre?
b) Draw a diagram to show the dimensions of the carton.
c) How much empty space is in the carton and cans once the 12 cans are placed in the carton?
d) What is the minimum amount of cardboard necessary to make this carton?

Chapter Problem Wrap-Up

You are to design a fountain for the garden of one of Emily’s customers.

• The fountain will have a cylindrical base with a cone on top.
• The cylindrical base will have a diameter of 1 m.
• The fountain is to be made of concrete.
• The entire fountain is to be coated with protective paint.

a) Make a sketch of your design, showing all dimensions.
b) How much concrete is needed to make the fountain?
c) What is the surface area that needs to be painted?
d) Concrete costs $100/m³. Each litre of protective paint costs $17.50 and covers 5 m². Find the total cost of the materials needed to make the fountain.